Optimal Solitaire Yahtzee* Strategies

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*) *Yahtzee* is a registered trademark of the Milton Bradley Company.

Micro Yahtzee: Equipment & Rules

- ONE fair die: values 1 [⚀] to 6 [⚅], equiprobable

- TWO primary categories on score card, each used once:

Category	Score
Double	value doubled
Square	value squared
TOTAL	sum above

- PLAY: roll, choose category, score, roll, score in other category

- AIM: maximize total score

Micro Yahtzee: Dilemma

- Empty score card:

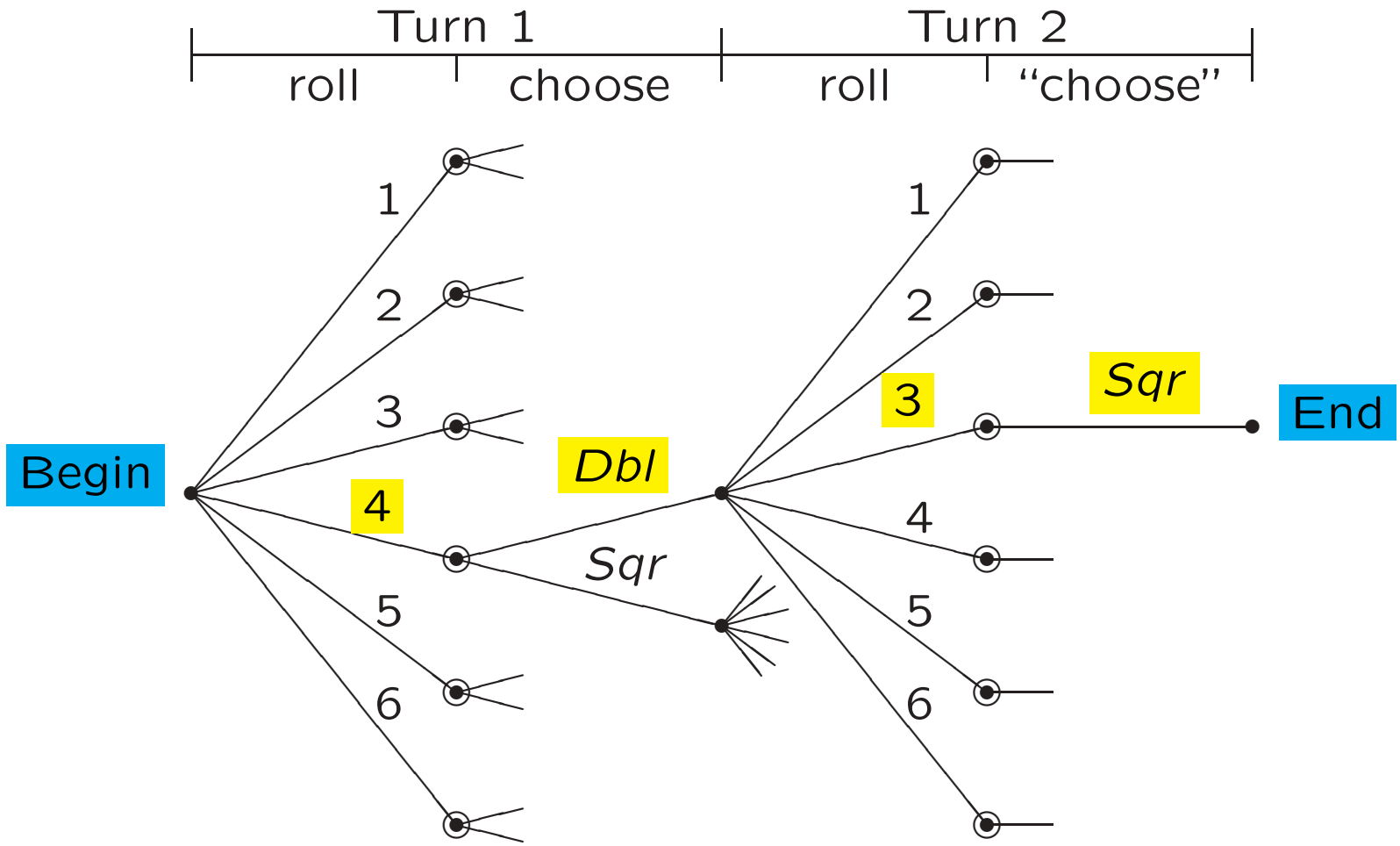
Category	Score
Double	. . .
Square	. . .
TOTAL	. . .

- What to do if first roll is 4?

  Score 8 in *Double*?

  Score 16 in *Square*?

Game Tree


Turn 1
Turn 2
roll
choose
roll
"choose"
Begin
1
2
3
4
5
6
Dbl
Sqr
1
2
3
Sqr
End
4
5
6

Micro Yahtzee: Analysis

Number of games: $6 \cdot 2 \cdot 6 \cdot 1 = 72$

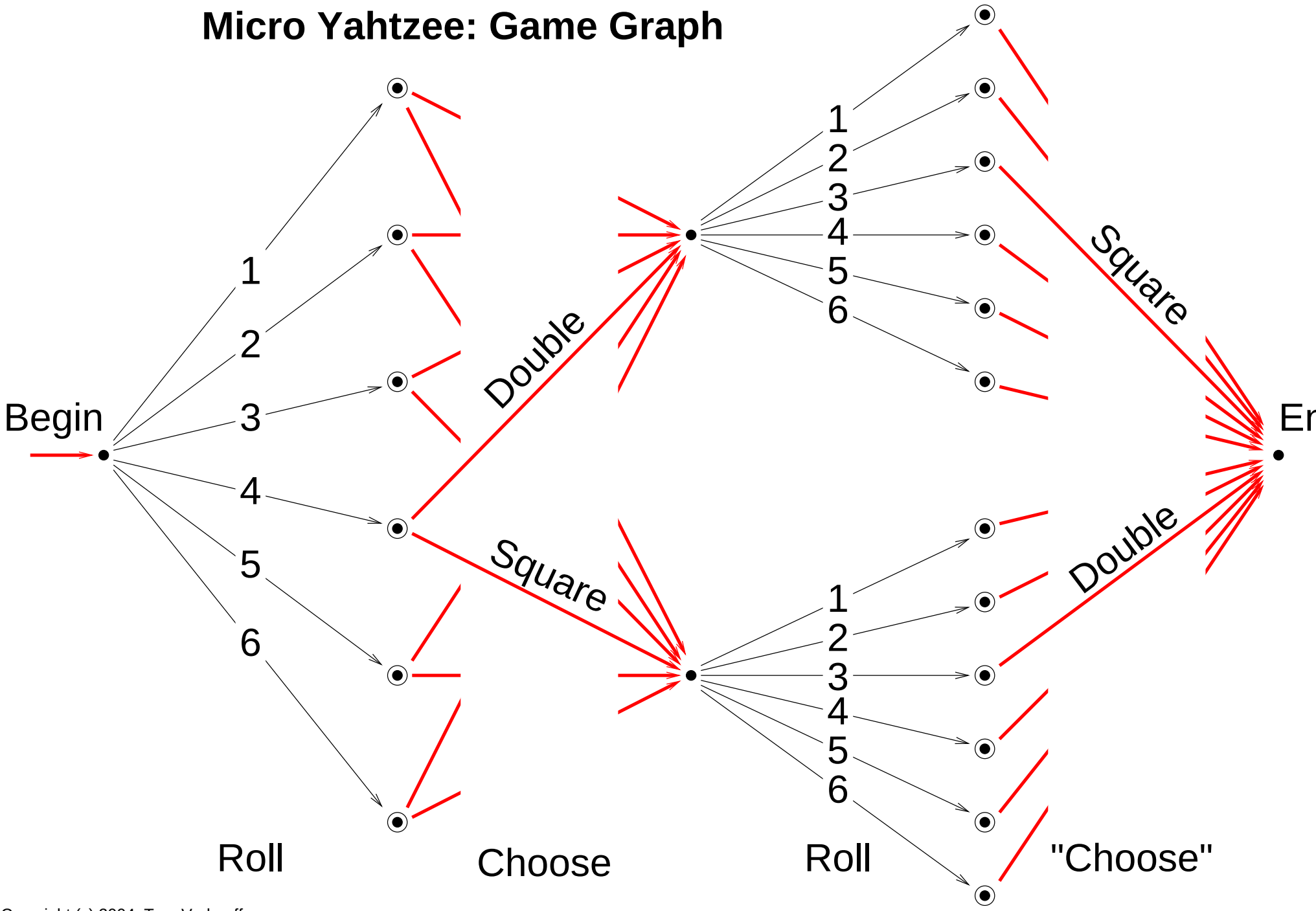
Number of deterministic strategies: $2^6 \cdot 1^6 = 64$

Probability of each game: $\frac{1}{6} \cdot \frac{1}{6} = 0.02777\ldots$

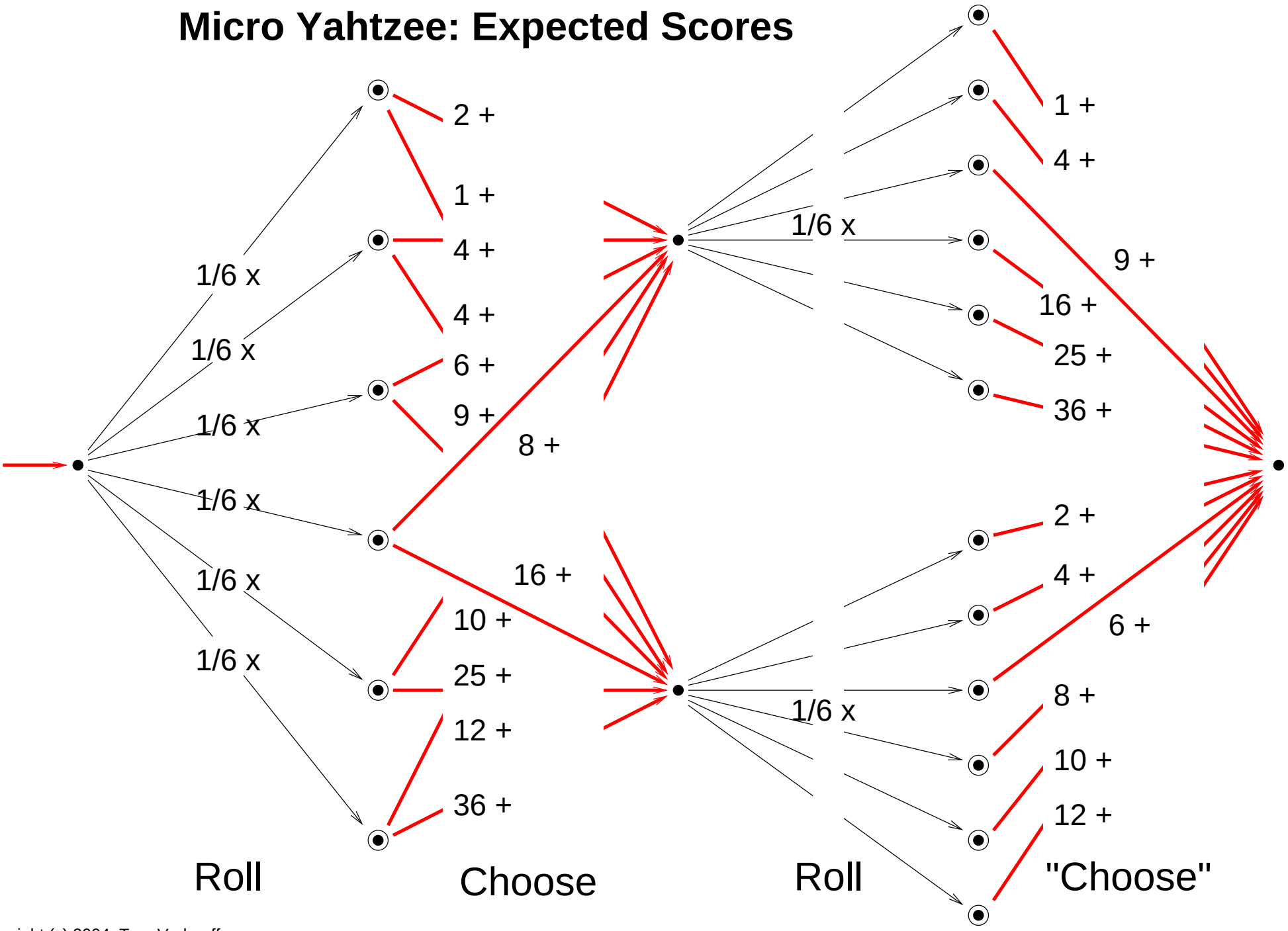
Solving Micro Yahtzee

- Dice rolls are *independent*: probability of game is *product*
- Event scores are *independent*: total score is *sum*
- Merge *equivalent* states $\longrightarrow$ Game Graph
- For each state, compute optimal expected additional score
- Start at the end (without choice), work backwards

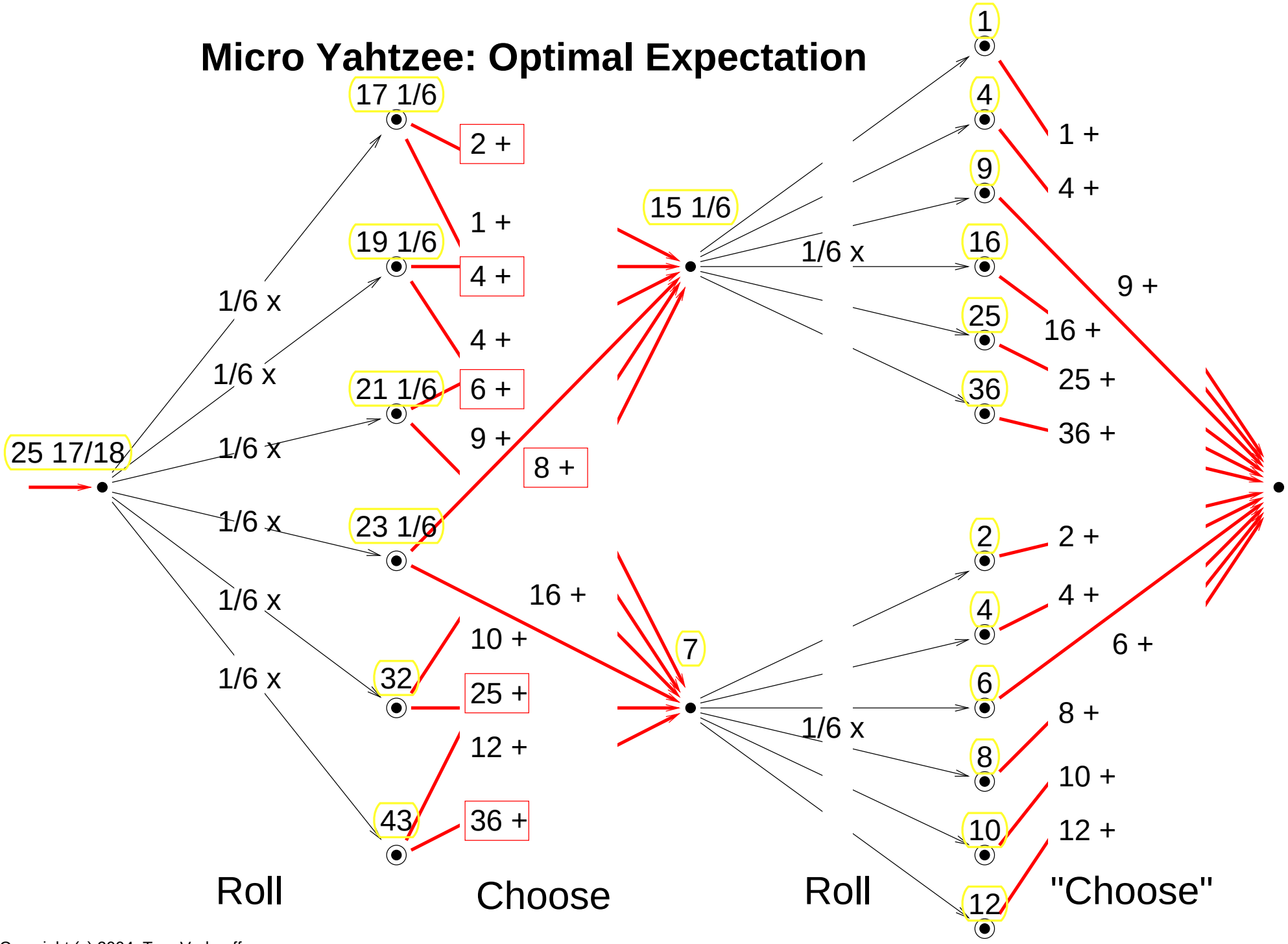
Micro Yahtzee: Game Graph



Micro Yahtzee: Expected Scores



Micro Yahtzee: Optimal Expectation



Yahtzee Equipment

- **5 Dice**: values 1 to 6, equiprobable
- **1 Score Card**: 13 primary categories

Yahtzee Score Card

Category	Score	
Aces	. . .	Upper Section
Twos	. . .	
Threes	. . .	
Fours	. . .	
Fives	. . .	
Sixes	. . .	
Upper Section Bonus	. . .	
Three of a Kind	. . .	Lower Section
Four of a Kind	. . .	
Full House	. . .	
Small Straight	. . .	
Large Straight	. . .	
Yahtzee	. . .	
Chance	. . .	
Extra Yahtzee Bonus	. . .	
GRAND TOTAL	. . .	

Official Yahtzee Playing Rules

Take *empty* score card

repeat

Roll *all* dice

Keep *any** dice, reroll other dice

Keep *any** dice, reroll other dice

Score roll in *any** *empty* primary category

until *all* primary categories scored

Calculate *GRAND TOTAL* for final score

*) Player is free to choose among options

Aim: Maximize final score

Official Yahtzee Scoring Rules

Category	Condition	Score
Aces	—	sum 1s
Twos	—	sum 2s
Threes	—	sum 3s
Fours	—	sum 4s
Fives	—	sum 5s
Sixes	—	sum 6s
U. S. Bonus	U.S.Tot ≥ 63	35 once
Three of a Kind	≥ 3 equals	sum values
Four of a Kind	≥ 4 equals	sum values
Full House	2+3 equals *	25
Small Straight	≥ 4 in seq. *	30
Large Straight	5 in seq. *	40
Yahtzee	5 equals	50
Chance	—	sum values
Extra Y. Bonus	5 equals & 50 at Y.	100 each
GRAND TOTAL	—	sum above

Joker Rule

*) 5 ys act here as Joker, *provided*

- Category ys in Upper Section has already been scored, and
- *Yahtzee* has already been scored

Dilemmas

- **First turn, first roll: 1 1 6 6 6** What to do?

  Keep 6 6 6?

  Keep all and score 25 in *Full House*?

- **First turn, second roll: 1 1 3 4 6** What to do?

- **First turn, third roll: 6 6 6 6 1** What to do?

  Score 24 in *Sixes*?

  Score 25 in *Four of a Kind*?

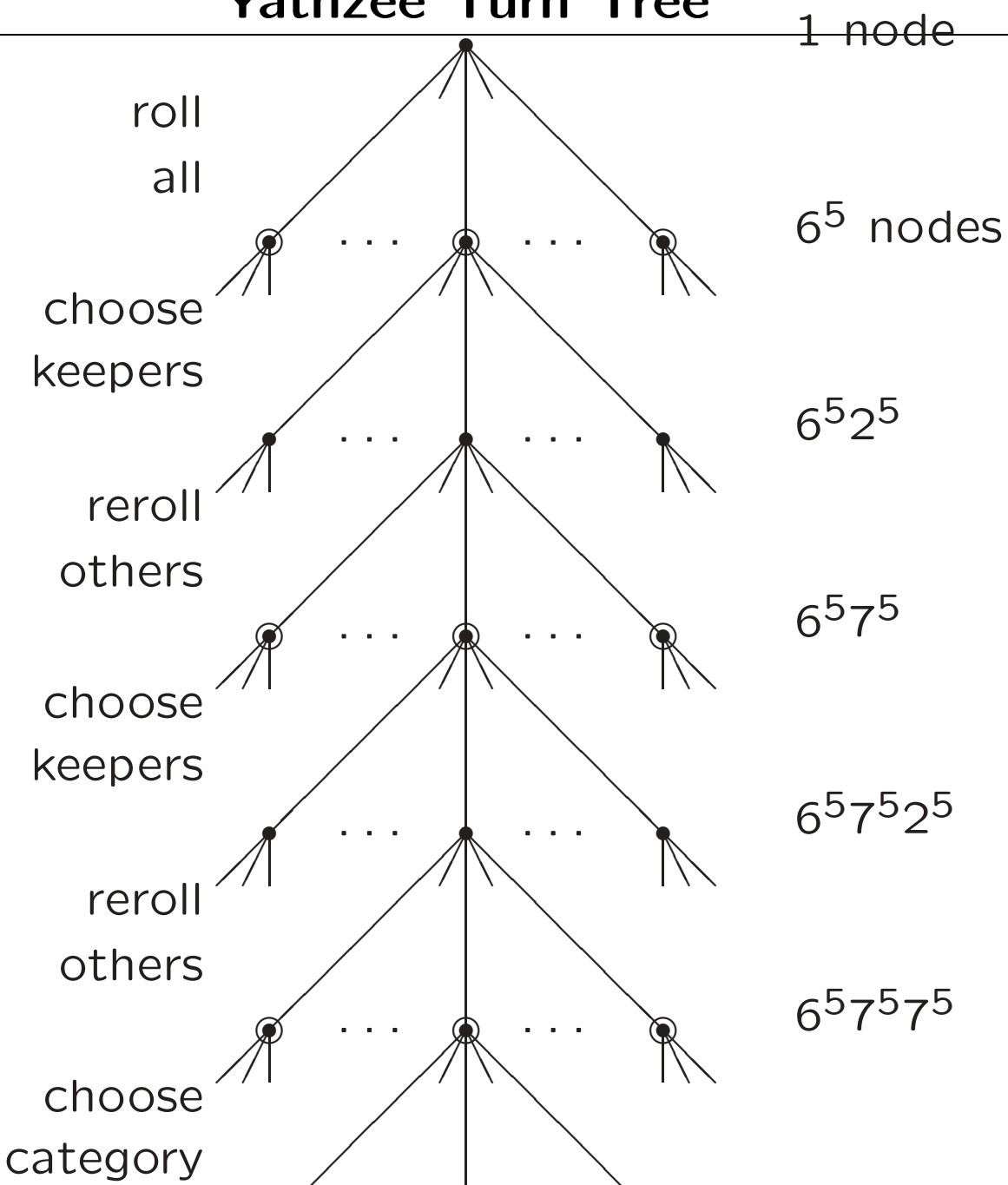
Random Play (without Bonuses and Jokers)

Category	Probability	Expected Score
Aces	1	0.83
Twos	1	1.67
Threes	1	2.50
Fours	1	3.33
Fives	1	4.17
Sixes	1	5.00
Three of a Kind	1656/7776	3.73
Four of a Kind	156/7776	0.35
Full House	300/7776	0.96
Small Straight	1200/7776	4.63
Large Straight	240/7776	1.23
Yahtzee	6/7776	0.04
Chance	1	17.50
GRAND TOTAL		45.95

Optimality Criteria

- Maximize **expected** final score
- Minimize **variance** in final score
- Maximize **probability to beat High Score**
- Maximize **probability to beat opponent**
- Maximize **minimum** final score

Yathzee Turn Tree



Yahtzee Game Tree

- **Number of games**:

$$\left(6^5 \cdot 7^5 \cdot 7^5\right) \cdot 13! \approx 1.7 \times 10^{170}$$

- **Game probabilities**: range from

$$\left(\left(6^{-5}\right)^3\right)^{13} \approx 5.5 \times 10^{-151}$$

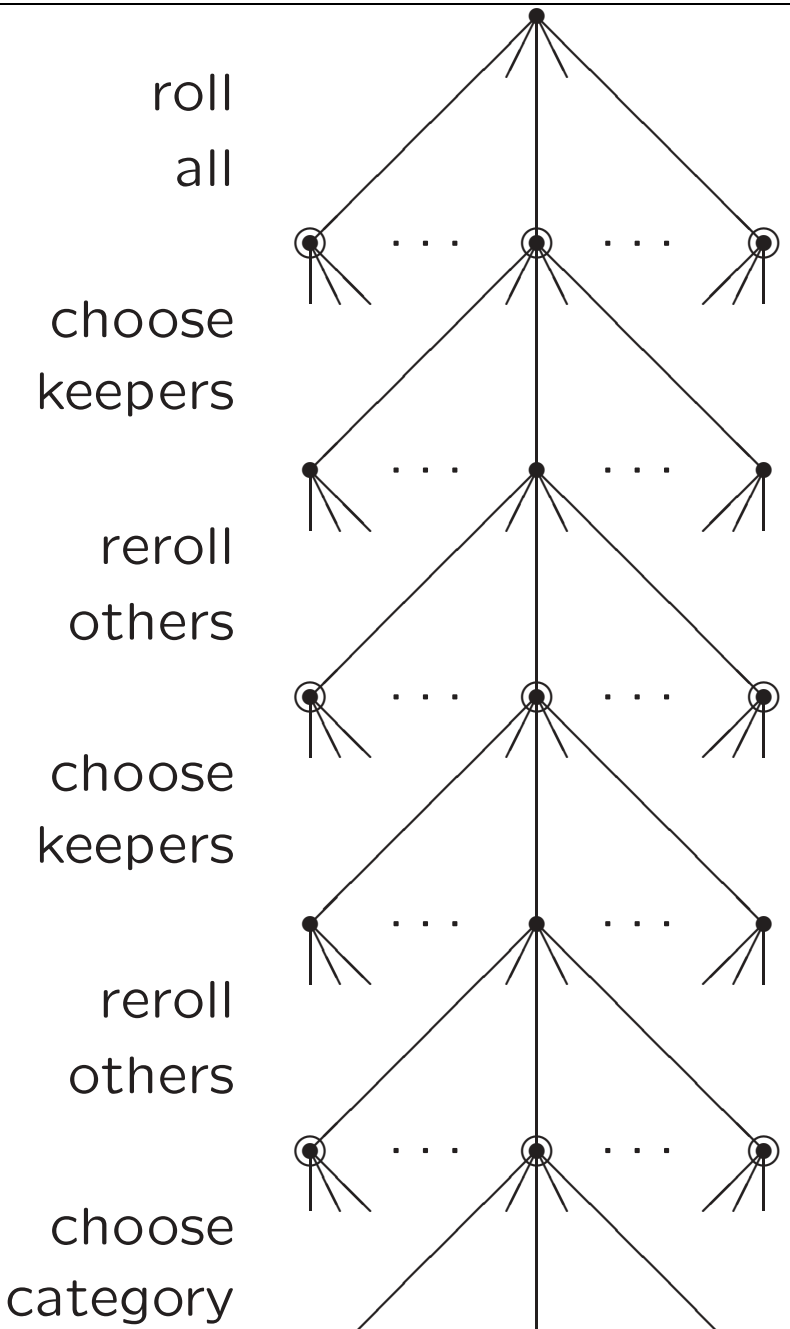
  to

$$\left(6^{-5}\right)^{13} \approx 3.8 \times 10^{-50}$$

- **Number of deterministic strategies**:

$$10^{10^{100}} \qquad ??$$

Reduced Yathzee Turn Graph



Reduced Yahtzee Game States

Between turns: $\boxed{2^{18} \cdot 3 = 786\,432}$

- Set of unscored primary categories: $\boxed{2^{13}}$
- How much needed for *Upper Section Bonus*:

  $\boxed{64 = 2^6}$ $(0\,..\,63)$
- Will 5 equals get *Extra Yahtzee Bonus*:

  $\boxed{2}$ (false, true; only when Yahtzee score box used)

Within turns: $\boxed{1 + 2 \cdot 462 + 3 \cdot 252 = 1681}$

Reduced Yahtzee Game States

- Roll index: $\boxed{3}$ $(1\,..\,3)$
- Roll versus Choose: $\boxed{2}$
- Rolled dice: $\boxed{\binom{5+6-1}{5} = 252}$

  Kept dice: $\boxed{\binom{5+7-1}{5} = 462}$

Computational Approach

- **Dynamic programming**

  Store optimal-expected-additional-scores to avoid recomputation

- **Two-level**

  Store optimal-expected-additional-score *between* turns only:
  Table of 786 432 reals of 8 byte = 6 MB

  Recompute *within turns*

- **Self-initializing**

  Compute required states only: 536 448

And now what?

- **Characterize** optimal strategies

  Exact results versus simulation

- **Compare** to other strategies

  E.g. random play

- Investigate effect of **rule changes**

  E.g. 2 or 4 rolls per turn, no Jokers, ...

What's in it for you?

- **Optimal Solitaire Yahtzee Player**

  Submit your game state on WWW and get advice

- **Yahtzee Proficiency Test**

  Play game on WWW and get analysis

- Also available as stand-alone PC program

Approximate Results

- **Numeric Evaluation**: Evalute exact recipe with finite precision

  Issues:

  - numeric stability (rounding, cancellation)
  - how many bits precision

- **Simulation**: Take average over a number of instances

  Issues:

  - quality of random number generator
  - how many instances (variance)

Dilemmas Resolved

- **First turn, first roll: 1 1 6 6 6**

Keep 6 6 6:	265.12 ± 61
Keep all and score 25 in *Full House*:	253.91 ± 57

- **First turn, second roll: 1 1 3 4 6**

Keep 3 4:	245.17 ± 57
Keep 1 1:	245.14 ± 57
Keep 4:	244.96 ± 57
Keep 3:	244.74 ± 57
Keep none:	244.55 ± 57
Keep 6:	244.52 ± 57

Dilemmas Resolved

- **First turn, third roll: 6 6 6 6 1**

  Score 24 in *Sixes*: 268.23 ± 53

  Score 25 in *Four of a Kind*: 260.54 ± 54

Optimal Strategy Trivia

- Expected final score: 254.59 ± 60
- Median final score: 248
- Best roll in first turn: $y\ y\ y\ y\ y$

  Score 50 in *Yahtzee*: 320.84 ± 83
- Worst first roll in first turn: 1 1 2 3 6

  Keep 6: 249.83 ± 58

Optimal Strategy Trivia

- Worst third roll in first turn: 2 3 4 4 6

  Score 19 in *Chance*: 238.96 ± 57

- Minimum score: 12

Final Scores per Category for Optimal Strategy

Category	E	SD	% 0
Aces	1.88	1.22	10.84
Twos	5.28	2.00	1.80
Threes	8.57	2.71	0.95
Fours	12.16	3.29	0.60
Fives	15.69	3.85	0.50
Sixes	19.19	4.64	0.53
U. S. Bonus	23.84	16.31	31.88
Three of a Kind	21.66	5.62	3.26
Four of a Kind	13.10	11.07	36.34
Full House	22.59	7.38	9.63
Small Straight	29.46	3.99	1.80
Large Straight	32.71	15.44	18.22
Yahtzee	16.87	23.64	66.26
Chance	22.01	2.54	0.00
Extra Y. Bonus	9.58	34.08	91.76
GRAND TOTAL	254.59	59.61	0.00
Yahtzees Rolled	0.46	0.69	63.24
Jokers Applied	0.04	0.19	96.30

Final Scores, playing w/o Extra Yahtzee Bonus and Jokers

Category	E	SD	% 0
Aces	1.82	1.14	9.19
Twos	5.25	1.95	1.31
Threes	8.57	2.65	0.59
Fours	12.19	3.24	0.46
Fives	15.74	3.81	0.40
Sixes	19.29	4.61	0.46
U. S. Bonus	24.14	16.19	31.02
Three of a Kind	22.23	5.50	3.44
Four of a Kind	13.04	11.44	39.38
Full House	22.86	6.99	8.54
Small Straight	29.53	3.71	1.55
Large Straight	33.04	15.16	17.40
Yahtzee	15.89	23.28	68.21
Chance	22.26	2.44	0.00
GRAND TOTAL	245.87	39.82	0.00
Yahtzees Rolled	0.41	0.61	64.76
Jokers Applied	–	–	–

Distribution of Final Score for Optimal Strategy

Score range	%	Cum.%	
100 – 119	0 %	0 %	
120 – 139	0 %	0 %	
140 – 159	2 %	2 %	
160 – 179	3 %	5 %	
180 – 199	9 %	14 %	
200 – 219	13 %	27 %	
220 – 239	14 %	41 %	
240 – 259	20 %	60 %	
260 – 279	19 %	80 %	
280 – 299	6 %	86 %	
300 – 319	5 %	90 %	
320 – 339	2 %	92 %	
340 – 359	1 %	93 %	
360 – 379	1 %	94 %	
380 – 399	2 %	96 %	
400 – 419	2 %	98 %	
420 – 439	1 %	99 %	
440 – 459	0 %	99 %	
460 – 479	0 %	99 %	
480 – 499	0 %	99 %	

Distribution, playing w/o Extra Yahtzee Bonus and Jokers

Score range	%	Cum.%	
100 – 119	0 %	0 %	
120 – 139	0 %	0 %	
140 – 159	2 %	2 %	
160 – 179	3 %	5 %	
180 – 199	9 %	14 %	
200 – 219	13 %	27 %	
220 – 239	14 %	40 %	
240 – 259	21 %	61 %	
260 – 279	21 %	82 %	
280 – 299	8 %	90 %	
300 – 319	6 %	97 %	
320 – 339	3 %	100 %	
340 – 359	0 %	100 %	

Results based on **simulation** of 10^5 games

Cumulative Distribution of Optimal Final Score

Final score f	**% Games scoring** $< f$
152	1 %
180	5 %
195	10 %
218	25 %
248	50 %
273	75 %
319	90 %
388	95 %
474	99 %

Results based on **simulation** of 10^6 games

Cumulative Distribution of Final Score

Without **Extra Yahtzee Bonus** and **Jokers**

Final score f	**% Games scoring** $< f$
152	1 %
180	5 %
195	10 %
218	25 %
248	50 %
271	75 %
299	90 %
317	95 %
327	99 %

Results based on **simulation** of 10^6 games

Earliest Scoring per Category for Optimal Strategy

Category	Earliest turn scoring	
	Non-Zero	Zero
Aces	1	2
Twos	1	3
Threes	1	4
Fours	1	5
Fives	1	6
Sixes	1	9
Three of a Kind	1	7
Four of a Kind	2	2
Full House	1	5
Small Straight	1	10
Large Straight	1	7
Yahtzee	1	3
Chance	1	never

Last Turn Values per Category (w/o Extra Yahtzee Bonus)

Category	E	SD	% 0
Aces	2.11	1.10	6.49
Twos	4.21	2.21	6.49
Threes	6.32	3.31	6.49
Fours	8.43	4.42	6.49
Fives	10.53	5.52	6.49
Sixes	12.64	6.62	6.49
Three of a Kind	15.19	10.42	28.76
Four of a Kind	5.61	9.66	72.26
Full House	9.15	12.04	63.39
Small Straight	18.48	14.59	38.40
Large Straight	10.61	17.66	73.47
Yahtzee	2.30	10.48	95.40
Chance	23.33	3.16	0.00

Game with Minimum Score of Optimal Strategy

Turn	Third Roll	Score	in Category
1	1 4 4 5 5	1	*Aces*
2	1 2 3 5 5	2	*Twos*
3	1 1 2 2 6	0	*Four of a Kind*
4	1 2 2 4 6	0	*Yahtzee*
5	1 1 2 2 6	0	*Threes*
6	1 2 2 3 3	0	*Fours*
7	1 2 2 3 3	0	*Fives*
8	1 2 2 3 3	0	*Full House*
9	1 2 2 3 3	0	*Sixes*
10	1 1 2 3 3	0	*Large Straight*
11	1 1 2 2 3	9	*Chance*
12	4 5 5 6 6	0	*Three of a Kind*
13	5 6 6 6 6	0	*Small Straight*
		12	*GRAND TOTAL*

Game of Optimal Strategy against Demonic Dice

Turn	Roll/Keep	Score	in Category
1	1 1 2 3 $\underline{6}$		
	1 $\underline{2}$ $\underline{3}$ $\underline{5}$ 6		
	1 2 3 5 6	1	*Aces*
2	1 1 2 3 $\underline{6}$		
	$\underline{1}$ $\underline{1}$ $\underline{1}$ 2 6		
	1 1 1 3 4	10	*Three of a Kind*
3	1 1 2 $\underline{3}$ 6		
	1 1 1 $\underline{3}$ 6		
	1 1 3 5 6	0	*Four of a Kind*
4	1 1 1 2 $\underline{6}$		
	1 1 1 $\underline{2}$ 6		
	1 2 3 5 5	2	*Twos*
5	1 1 2 2 $\underline{6}$		
	1 2 2 2 $\underline{6}$		
	2 2 2 4 6	0	*Yahtzee*
6	1 2 2 2 2		
	1 2 2 2 2		
	1 2 2 2 6	0	*Threes*
7	2 3 3 3 3		
	2 3 3 3 3		
	2 2 2 2 3	0	*Fours*

Game against Demonic Dice (cont'd)

Turn	Roll/Keep	Score	in Category
8	1 2 2 2 2		
	2 3 3 3 3		
	2 2 2 2 3	0	*Fives*
9	1 2 2 2 2		
	1 2 2 2 2		
	2 2 2 3 4	0	*Full House*
10	2 2 5 5 5		
	2 3 3 3 3		
	2 2 2 2 3	0	*Sixes*
11	2 2 2 6 6		
	1 2 2 2 2		
	2 2 2 2 3	0	*Large Straight*
12	2 2 2 6 6		
	1 2 2 2 2		
	1 1 1 1 2	6	*Chance*
13	5 6 6 6 6		
	5 5 6 6 6		
	5 6 6 6 6	0	*Small Straight*
		19	*GRAND TOTAL*

Remaining Challenges

- Best strategy to beat given *High Score*

  Approximation via normal distribution and computed mean & variance

  Optimal premature stopping

- Best strategy for *Group Yahtzee*

  Approximation

References

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