

Optimal Solitaire Yahtzee* Strategies

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<http://wwpa.win.tue.nl/misc/yahtzee/>

*) *Yahtzee* is a registered trademark of
the Milton Bradley Company.

Equipment

- **5 Dice:** values 1 through 6 equiprobable
- **1 Score Card:**

Category	Score	
<i>Aces*</i>	...	S
<i>Twos*</i>	...	U
<i>Threes*</i>	...	P
<i>Fours*</i>	...	P
<i>Fives*</i>	...	E
<i>Sixes*</i>	...	R
<i>Upper Section Bonus</i>	...	N
<i>Three of a Kind*</i>	...	S
<i>Four of a Kind*</i>	...	L
<i>Full House*</i>	...	O
<i>Small Straight*</i>	...	W
<i>Large Straight*</i>	...	E
<i>Yahtzee*</i>	...	R
<i>Chance*</i>	...	O
<i>Extra Yahtzee Bonus</i>	...	N
GRAND TOTAL	...	

*) Primary categories

Playing Rules

Take *empty* score card

repeat

Roll *all* dice

Keep any* dice, reroll other dice

Keep any* dice, reroll other dice

Score roll in any* *empty* primary category

until *all* primary categories scored

Calculate *GRAND TOTAL* for final score

Aim: Maximize final score

*) Player is free to choose among options

Scoring Rules

Category	Condition	Score
<i>Aces</i>	—	sum 1s
<i>Twos</i>	—	sum 2s
<i>Threes</i>	—	sum 3s
<i>Fours</i>	—	sum 4s
<i>Fives</i>	—	sum 5s
<i>Sixes</i>	—	sum 6s
<i>U. S. Bonus</i>	U.S.Tot \geq 63	35 once
<i>Three of a Kind</i>	\geq 3 equals	sum values
<i>Four of a Kind</i>	\geq 4 equals	sum values
<i>Full House</i>	2+3 equals*	25
<i>Small Straight</i>	\geq 4 in seq.*	30
<i>Large Straight</i>	5 in seq.*	40
<i>Yahtzee</i>	5 equals	50
<i>Chance</i>	—	sum values
<i>Extra Y. Bonus</i>	5 equals & 50 at Y.	100 each
<i>GRAND TOTAL</i>	—	sum above

*) 5 *ys* act here as **Joker**, *provided* categories *ys* and *Yahtzee* have been scored already.

Dilemmas

- **First turn, first roll: 1 1 6 6 6**

What to do?

Keep 6 6 6?

Keep all and score 25 in *Full House*?

- **First turn, second roll: 1 1 3 4 6**

What to do?

- **First turn, third roll: 6 6 6 6 1**

What to do?

Score 24 in *Sixes*?

Score 25 in *Four of a Kind*?

Random Play

Without Bonuses and Jokers

Category	Probability	Expected Score
<i>Aces</i>	1	0.83
<i>Twos</i>	1	1.67
<i>Threes</i>	1	2.50
<i>Fours</i>	1	3.33
<i>Fives</i>	1	4.17
<i>Sixes</i>	1	5.00
<i>Three of a Kind</i>	1656/7776	3.73
<i>Four of a Kind</i>	156/7776	0.35
<i>Full House</i>	300/7776	0.96
<i>Small Straight</i>	1200/7776	4.63
<i>Large Straight</i>	240/7776	1.23
<i>Yahtzee</i>	6/7776	0.04
<i>Chance</i>	1	17.50
GRAND TOTAL		45.95

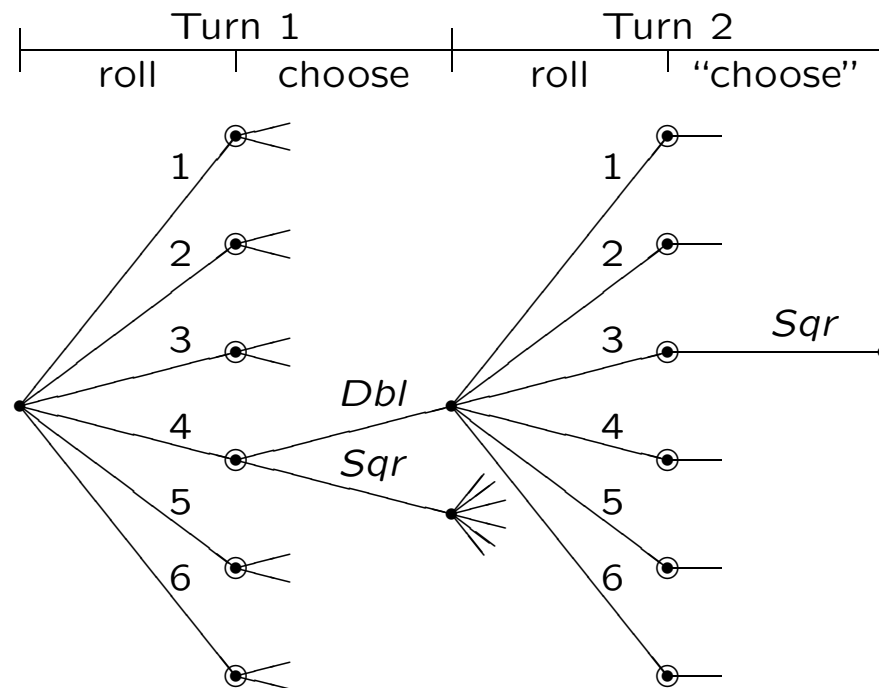
Micro Yahtzee

- ONE die
- NO keeping and rerolling
- TWO primary categories:

Category	Score
<i>Double</i>	value doubled
<i>Square</i>	value squared
<i>TOTAL</i>	sum above

- How to maximize final score?
- What to do if first roll is 4?
Score 8 in *Double*?
Score 16 in *Square*?

Game Tree



Choice states are circled: ●

$$\# \text{ Games: } 6 \cdot 2 \cdot 6 \cdot 1 = 72$$

$$\# \text{ Deterministic strategies: } 2^6 \cdot 1^6 = 64$$

Markov Decision Processes (sort of :-)

- **State space** $S = R \uplus C$

The MDP is always in one state of S .

- **Initial state** I with $I \in S$

- **Event sets** $E.s$ for $s \in S$

In state s , one event from $E.s$ occurs.
Terminate if $E.s = \emptyset$.

- **Event probabilities** $p.s$ for $s \in S$

Event $e \in E.s$ occurs with probability $p.s.e$.

$$\sum_{e \in E.s} p.s.e = 1$$

- **Event scores** $f.s$ for $s \in S$

Event $e \in E.s$ scores $f.s.e$.

- **Transition function** (juxtaposition)

Event $e \in E.s$ leads to next state se .

Yahtzee as MDP

- **State space** $S = R \uplus C$

R : roll states; C : choice states.

- **Event sets** $E.s$ for $s \in S$

Roll outcomes for $s \in R$.

Keep or score choices for $s \in C$.

- **Event scores** $f.s$ for $s \in S$

$f.s.e = 0$ for $s \in R$.

$f.s.e \geq 0$ for $s \in C$.

- **Transition function**

R and C states **alternate**.

Acyclic.

Markov Decision Strategies

- **Decision strategy** D defines $p.s$ for $s \in C$

Deterministic if $p.s.e \in \{0, 1\}$

- **Game** g after state s :

Sequence of successive events starting in s

Resulting state sg : $s\langle \rangle = s$, $s(eg) = (se)g$

- Set $G.s$ of **complete games** after s :

$$G.s = \{g \mid E.sg = \emptyset\}$$

- **Score** $F.s.g$ of game g after s :

$$F.s.\langle \rangle = 0$$

$$F.s.eg = f.s.e + F.se.g$$

- **Probability** $P.s.g$ of game g after s :

$$P.s.\langle \rangle = 1$$

$$P.s.eg = p.s.e * P.se.g$$

Optimality Criteria

- Maximize **expected** final score
- Minimize **variance** in final score
- Maximize **probability to beat High Score**
- Maximize **probability to beat opponent**
- Maximize **minimum** final score

Optimal Strategies

- Expected final score \mathcal{E}_D by strategy D :

$$\mathcal{E}_D = \sum_{g \in G.I} P.I.g * F.I.g$$

- Optimal strategy achieves

$$\hat{\mathcal{E}} = \max_D \mathcal{E}_D$$

- Conditional expectation after state s :

$$\mathcal{E}.s = \sum_{g \in G.s} P.s.g * F.s.g$$

- Recurrence relations:

$$\mathcal{E}.s = \sum_{e \in E.s} p.s.e * (f.s.e + \mathcal{E}.se)$$

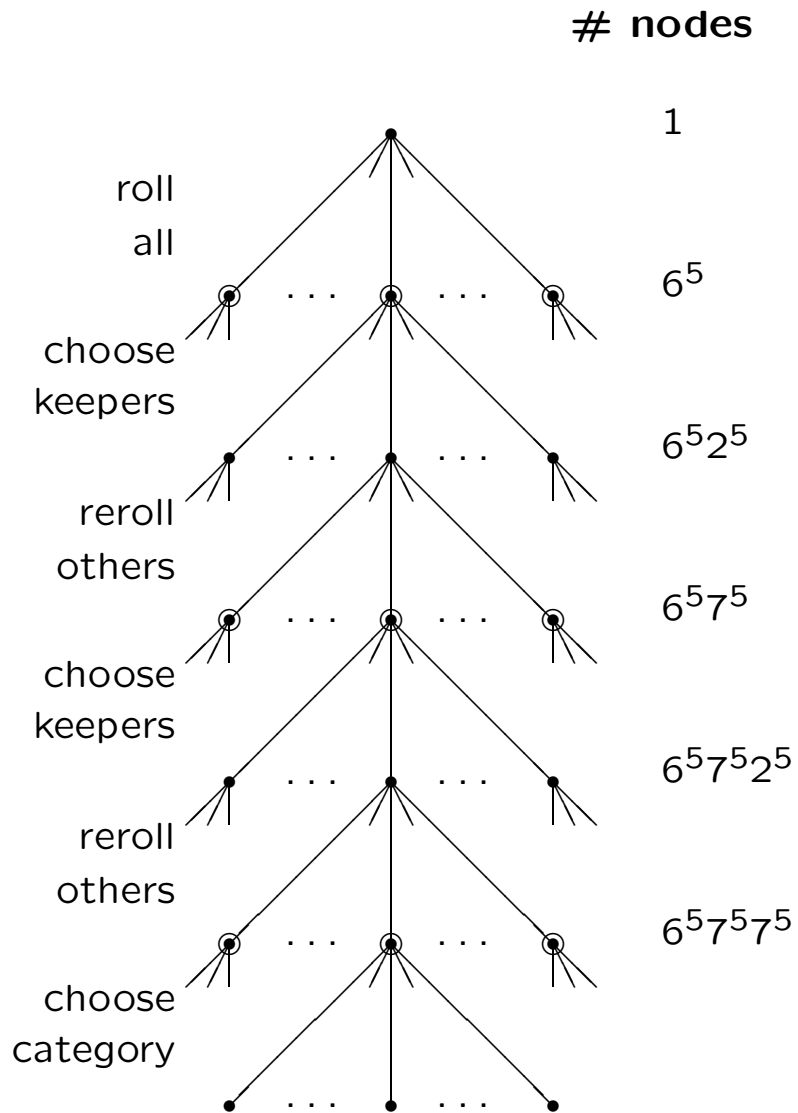
$$\hat{\mathcal{E}}.s = \begin{cases} \sum_{e \in E.s} p.s.e * \hat{\mathcal{E}}.se & \text{for } s \in R \\ \max_{e \in E.s} (f.s.e + \hat{\mathcal{E}}.se) & \text{for } s \in C \end{cases}$$

Recurrence Relation for \mathcal{E}

For $E.s \neq \emptyset$:

$$\begin{aligned} & \mathcal{E}.s \\ = & \{ \text{definition of } \mathcal{E}.s \} \\ & \sum_{g \in G.s} P.s.g * F.s.g \\ = & \{ g = eh \text{ with } e \in E.s \neq \emptyset, h \in G.se \} \\ & \sum_{e \in E.s} \sum_{h \in G.se} P.s.eh * F.s.eh \\ = & \{ \text{recurrences for } P.s.eh, F.s.eh \} \\ & \sum_{e \in E.s} \sum_{h \in G.se} p.s.e * P.se.h * (f.s.e + F.se.h) \\ = & \{ \text{distribution: } p.s.e \text{ independent of } h \} \\ & \sum_{e \in E.s} p.s.e * \sum_{h \in G.se} P.se.h * (f.s.e + F.se.h) \\ = & \{ \sum_{y \in G.x} P.x.y = 1 \} \\ & \sum_{e \in E.s} p.s.e * \left(f.s.e + \sum_{h \in G.se} P.se.h * F.se.h \right) \\ = & \{ \text{definition of } \mathcal{E}.se \} \\ & \sum_{e \in E.s} p.s.e * (f.s.e + \mathcal{E}.se) \end{aligned}$$

Yathzee Turn Tree



Yahtzee Game Tree

- **# Games:**

$$(6^5 \cdot 7^5 \cdot 7^5) \cdot 13! \approx 1.7 \times 10^{170}$$

- **Probabilities:** range from

$$\left((6^{-5})^3 \right)^{13} \approx 5.5 \times 10^{-151}$$

to

$$(6^{-5})^{13} \approx 3.8 \times 10^{-50}$$

- **# Strategies:**

$$10^{10^{100}} \quad ??$$

Reducing the State Space

- **State equivalence relation:**

$$s \sim t \Leftrightarrow F.s = F.t \wedge P.s = P.t$$

Equal future (past ignored)

- **Theorem** For equivalent states $s \sim t$:

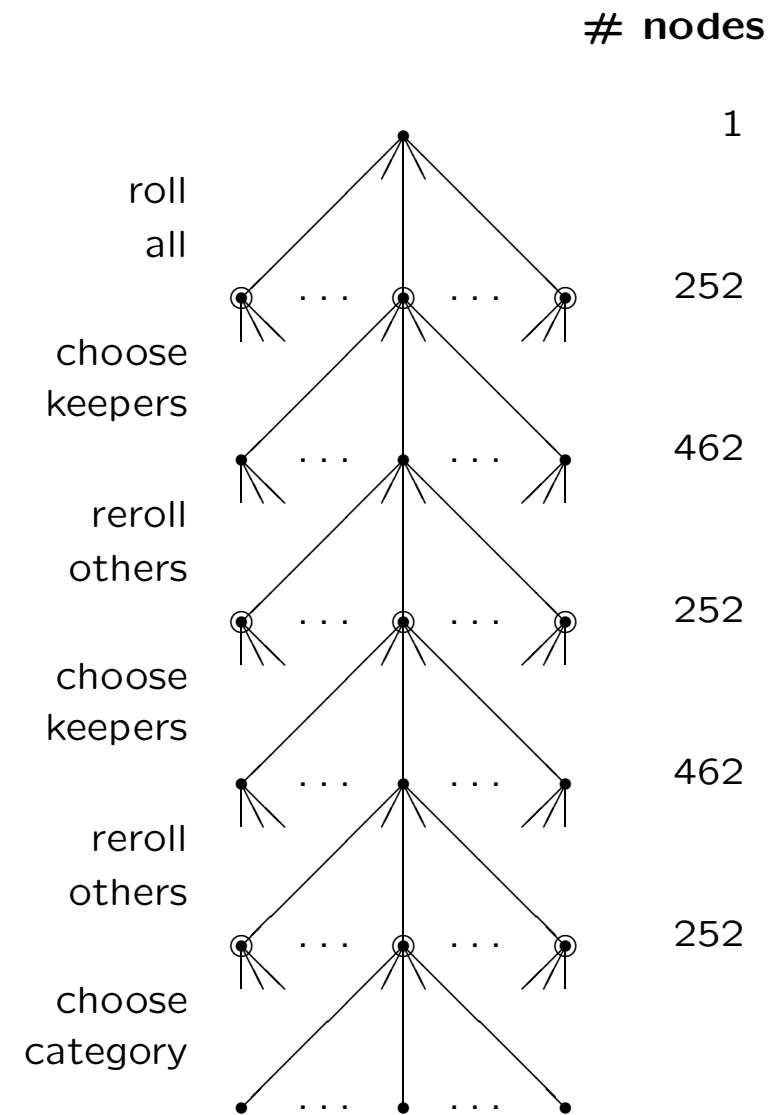
$$\mathcal{E}.s = \mathcal{E}.t$$

$$\hat{\mathcal{E}}.s = \hat{\mathcal{E}}.t$$

- **Reduced game tree:**

Merge equivalent states

Reduced Yahtzee Turn Graph



Reduced Yahtzee Game States

Between turns: $2^{18} \cdot 3 = 786\,432$

- Set of unscored primary categories: 2^{13}
- How much needed for *Upper Section Bonus*:
 $64 = 2^6$ (0..63)
- Will 5 equals get *Extra Yahtzee Bonus*:
 2 (false, true)

Within turns: $1 + 2 \cdot 462 + 3 \cdot 252 = 1681$

- Roll index: 3 (1..3)
- Roll versus Choose: 2
- Rolled dice: $\binom{5+6-1}{5} = 252$
- Kept dice: $\binom{5+7-1}{5} = 462$

Computational Approach

- **Dynamic programming**

Store $\hat{\mathcal{E}}.s$ to avoid recomputation

- **Two-level**

Store $\hat{\mathcal{E}}.s$ *between* turns only:

Table of 786 432 reals of 8 byte = 6 MB

Recompute *within turns*

- **Self-initializing**

Compute required states only: 536 448

And now what?

- **Characterize** optimal strategies
Exact results versus simulation
- **Compare** to other strategies
E.g. random play
- Investigate effect of **rule changes**
E.g. 2 or 4 rolls per turn, no Jokers, ...
- **Optimal Solitaire Yahtzee Player**
Submit your game state on WWW and get advice
- **Yahtzee Proficiency Test**
Play game on WWW and get analysis

Approximate Results

- **Numeric Evaluation:**
Evaluate exact recipe with finite precision
Issues:
 - numeric stability (rounding, cancellation)
 - how many bits precision
- **Simulation:**
Take average over a number of instances
Issues:
 - quality of random number generator
 - how many instances (variance)

Dilemmas Resolved

- **First turn, first roll: 1 1 6 6 6**

Keep 6 6 6: 265.12 ± 61

Keep all and
score 25 in *Full House*: 253.91 ± 57

- **First turn, second roll: 1 1 3 4 6**

Keep 3 4: 245.17 ± 57

Keep 1 1: 245.14 ± 57

Keep 4: 244.96 ± 57

Keep 3: 244.74 ± 57

Keep none: 244.55 ± 57

Keep 6: 244.52 ± 57

- **First turn, third roll: 6 6 6 6 1**

Score 24 in *Sixes*: 268.23 ± 53

Score 25 in *Four of a Kind*: 260.54 ± 54

Optimal Strategy Trivia

- Expected final score: 254.59 ± 60

- Median final score: 248

- Best roll in first turn: *y y y y y*

Score 50 in *Yahtzee*: 320.84 ± 83

- Worst first roll in first turn: 1 1 2 3 6

Keep 6: 249.83 ± 58

- Worst third roll in first turn: 2 3 4 4 6

Score 19 in *Chance*: 238.96 ± 57

- Minimum score: 12

Final Scores per Category

Optimal Solitaire Yahtzee Strategy

Category	E	SD	% 0
<i>Aces</i>	1.88	1.22	10.84
<i>Twos</i>	5.28	2.00	1.80
<i>Threes</i>	8.57	2.71	0.95
<i>Fours</i>	12.16	3.29	0.60
<i>Fives</i>	15.69	3.85	0.50
<i>Sixes</i>	19.19	4.64	0.53
<i>U. S. Bonus</i>	23.84	16.31	31.88
<i>Three of a Kind</i>	21.66	5.62	3.26
<i>Four of a Kind</i>	13.10	11.07	36.34
<i>Full House</i>	22.59	7.38	9.63
<i>Small Straight</i>	29.46	3.99	1.80
<i>Large Straight</i>	32.71	15.44	18.22
<i>Yahtzee</i>	16.87	23.64	66.26
<i>Chance</i>	22.01	2.54	0.00
<i>Extra Y. Bonus</i>	9.58	34.08	91.76
GRAND TOTAL	254.59	59.61	0.00
<i>Yahtzees Rolled</i>	0.46	0.69	63.24
<i>Jokers Applied</i>	0.04	0.19	96.30

Final Scores per Category

Without Extra Yahtzee Bonus and Jokers

Category	E	SD	% 0
<i>Aces</i>	1.82	1.14	9.19
<i>Twos</i>	5.25	1.95	1.31
<i>Threes</i>	8.57	2.65	0.59
<i>Fours</i>	12.19	3.24	0.46
<i>Fives</i>	15.74	3.81	0.40
<i>Sixes</i>	19.29	4.61	0.46
<i>U. S. Bonus</i>	24.14	16.19	31.02
<i>Three of a Kind</i>	22.23	5.50	3.44
<i>Four of a Kind</i>	13.04	11.44	39.38
<i>Full House</i>	22.86	6.99	8.54
<i>Small Straight</i>	29.53	3.71	1.55
<i>Large Straight</i>	33.04	15.16	17.40
<i>Yahtzee</i>	15.89	23.28	68.21
<i>Chance</i>	22.26	2.44	0.00
GRAND TOTAL	245.87	39.82	0.00
<i>Yahtzees Rolled</i>	0.41	0.61	64.76
<i>Jokers Applied</i>	–	–	–

Distribution of Final Score

Optimal Solitaire Yahtzee Strategy

Score range	%	Cum.%	
100 – 119	0 %	0 %	
120 – 139	0 %	0 %	
140 – 159	2 %	2 %	■
160 – 179	3 %	5 %	■
180 – 199	9 %	14 %	■
200 – 219	13 %	27 %	■
220 – 239	14 %	41 %	■
240 – 259	20 %	60 %	■
260 – 279	19 %	80 %	■
280 – 299	6 %	86 %	■
300 – 319	5 %	90 %	■
320 – 339	2 %	92 %	■
340 – 359	1 %	93 %	■
360 – 379	1 %	94 %	■
380 – 399	2 %	96 %	■
400 – 419	2 %	98 %	■
420 – 439	1 %	99 %	■
440 – 459	0 %	99 %	
460 – 479	0 %	99 %	
480 – 499	0 %	99 %	

Results based on **simulation** of 10^5 games

Distribution of Final Score

Without **Extra Yahtzee Bonus** and **Jokers**

Score range	%	Cum.%	
100 – 119	0 %	0 %	
120 – 139	0 %	0 %	
140 – 159	2 %	2 %	■
160 – 179	3 %	5 %	■
180 – 199	9 %	14 %	■
200 – 219	13 %	27 %	■
220 – 239	14 %	40 %	■
240 – 259	21 %	61 %	■
260 – 279	21 %	82 %	■
280 – 299	8 %	90 %	■
300 – 319	6 %	97 %	■
320 – 339	3 %	100 %	■
340 – 359	0 %	100 %	

Results based on **simulation** of 10^5 games

Cumulative Distribution of Final Score

Optimal Solitaire Yahtzee Strategy

Final score <i>f</i>	% Games scoring < <i>f</i>
152	1 %
180	5 %
195	10 %
218	25 %
248	50 %
273	75 %
319	90 %
388	95 %
474	99 %

Results based on **simulation** of 10^6 games

Cumulative Distribution of Final Score

Without **Extra Yahtzee Bonus** and **Jokers**

Final score <i>f</i>	% Games scoring < <i>f</i>
152	1 %
180	5 %
195	10 %
218	25 %
248	50 %
271	75 %
299	90 %
317	95 %
327	99 %

Results based on **simulation** of 10^6 games

Earliest Scoring per Category

Optimal Solitaire Yahtzee Strategy

Category	Earliest turn scoring	
	Non-Zero	Zero
<i>Aces</i>	1	2
<i>Twos</i>	1	3
<i>Threes</i>	1	4
<i>Fours</i>	1	5
<i>Fives</i>	1	6
<i>Sixes</i>	1	9
<i>Three of a Kind</i>	1	7
<i>Four of a Kind</i>	2	2
<i>Full House</i>	1	5
<i>Small Straight</i>	1	10
<i>Large Straight</i>	1	7
<i>Yahtzee</i>	1	3
<i>Chance</i>	1	never

Last Turn Values per Category

Without Extra Yahtzee Bonus

Category	E	SD	% 0
<i>Aces</i>	2.11	1.10	6.49
<i>Twos</i>	4.21	2.21	6.49
<i>Threes</i>	6.32	3.31	6.49
<i>Fours</i>	8.43	4.42	6.49
<i>Fives</i>	10.53	5.52	6.49
<i>Sixes</i>	12.64	6.62	6.49
<i>Three of a Kind</i>	15.19	10.42	28.76
<i>Four of a Kind</i>	5.61	9.66	72.26
<i>Full House</i>	9.15	12.04	63.39
<i>Small Straight</i>	18.48	14.59	38.40
<i>Large Straight</i>	10.61	17.66	73.47
<i>Yahtzee</i>	2.30	10.48	95.40
<i>Chance</i>	23.33	3.16	0.00

Game with Minimum Score

Optimal Solitaire Yahtzee Strategy

Turn	Third Roll	Score	in Category
1	1 4 4 5 5	1	<i>Aces</i>
2	1 2 3 5 5	2	<i>Twos</i>
3	1 1 2 2 6	0	<i>Four of a Kind</i>
4	1 2 2 4 6	0	<i>Yahtzee</i>
5	1 1 2 2 6	0	<i>Threes</i>
6	1 2 2 3 3	0	<i>Fours</i>
7	1 2 2 3 3	0	<i>Fives</i>
8	1 2 2 3 3	0	<i>Full House</i>
9	1 2 2 3 3	0	<i>Sixes</i>
10	1 1 2 3 3	0	<i>Large Straight</i>
11	1 1 2 2 3	9	<i>Chance</i>
12	4 5 5 6 6	0	<i>Three of a Kind</i>
13	5 6 6 6 6	0	<i>Small Straight</i>
		12	GRAND TOTAL

Game against Demonic Dice

Optimal Solitaire Yahtzee Strategy

Turn	Roll/Keep	Score	in Category
1	1 1 2 3 <u>6</u> 1 <u>2</u> <u>3</u> <u>5</u> 6 1 2 3 5 6	1	<i>Aces</i>
2	1 1 2 3 <u>6</u> <u>1</u> <u>1</u> <u>1</u> 2 6 1 1 1 3 4	10	<i>Three of a Kind</i>
3	1 1 2 <u>3</u> 6 1 1 1 <u>3</u> 6 1 1 3 5 6	0	<i>Four of a Kind</i>
4	1 1 1 2 <u>6</u> 1 1 1 <u>2</u> 6 1 2 3 5 5	2	<i>Twos</i>
5	1 1 2 2 <u>6</u> 1 2 2 2 <u>6</u> 2 2 2 4 6	0	<i>Yahtzee</i>
6	1 2 2 2 2 1 2 2 2 2 1 2 2 2 6	0	<i>Threes</i>
7	2 3 3 3 3 2 3 3 3 3 2 2 2 2 3	0	<i>Fours</i>

Game against Demonic Dice (cont'd)

Turn	Roll/Keep	Score	in Category
8	1 2 2 2 2 2 3 3 3 3 2 2 2 2 3	0	<i>Fives</i>
9	1 2 2 2 2 1 2 2 2 2 2 2 2 3 4	0	<i>Full House</i>
10	2 2 5 5 5 2 3 3 3 3 2 2 2 2 3	0	<i>Sixes</i>
11	2 2 2 6 6 1 2 2 2 2 2 2 2 2 3	0	<i>Large Straight</i>
12	2 2 2 6 6 1 2 2 2 2 1 1 1 1 2	6	<i>Chance</i>
13	5 6 6 6 6 5 5 6 6 6 5 6 6 6 6	0	<i>Small Straight</i>
		19	<i>GRAND TOTAL</i>

Remaining Challenges

- Best strategy to beat given *High Score*
Approximation via normal distribution and computed mean & variance
Optimal premature stopping
- Best strategy for *Group Yahtzee*
Approximation